CS271 Homework 6 Solution

7-1-10
We need to compute the number of poker hands that contain the two of diamonds and the three of spades. There is no choice about choosing these two cards. To form the rest of the hand, we need to choose 3 cards from the 50 remaining cards, so there are C(50, 3) hands containing these two specific cards. Therefore the answer to the question is the ratio $\frac{C(50,3)}{C(52,5)} = \frac{5}{663} = 0.0075$.

7-1-18
There are clearly only $10 \times 4 = 40$ straight flushes, since all we get to specify for a straight flush is the starting (lowest) kind in the straight (anything from ace up to ten) and the suit. Therefore the answer is $40/C(52, 5) = 40/2598960 = 1/64974$.

7-1-30
In order to specify a winning ticket, we must choose five of the six numbers to match ($C(6, 5) = 6$ ways to do so) and one number from among the remaining 34 numbers not to match ($C(34, 1) = 34$ ways to do so). Therefore there are $6 \times 34 = 204$ winning tickets. Since there are $C(40, 6) = 3838380$ tickets in all, the answer is $204/3838380 = 5.310^{-5}$, or about 1 chance in 19000.

7-2-10
Note that there are 26! permutations of the letters, so the denominator in all of our answers is 26!. To find the numerator, we have to count the number of ways that the given event can happen. Alternatively, in some cases we may be able to exploit symmetry.
a) There are 13! possible arrangements of the first 13 letters of the permutation, and in only one of these are they in alphabetical order. Therefore the answer is 1/13!.
b) Once these two conditions are met, there are 24! ways to choose the remaining letters for positions 2 through 25. Therefore the answer is $24!/26! = 1/650$.
c) In effect we are forming a permutation of 25 itemsthe letters b through y and the double letter combination az or za. There are 25! ways to permute these items, and for each of these permutations there are two choices as to whether a or z comes first. Thus there are $2 \times 25!$ ways for form such a permutation, and therefore the answer is $2 \times 25!/26! = 1/13$.
d) By part (c), the probability that a and b are next to each other is 1/13. Therefore the probability that a and b are not next to each other is 12/13.
e) There are six ways this can happen: ax24z , zx24a, xax23z , xzx23a, ax23zx, and zx23ax, where x stands for any letter other than a and z (but of course all the xs are different in each permutation). In each of these there are 24! ways to permute the letters other than a and z , so there are 24! permutations of each type. This gives a total of $6 \times 24!$ permutations meeting the conditions, so the answer is $(6 \times 24!)/26! = 3/325$.
f) Looking at the relative placements of z , a, and b, we see that one third of the time, z will come first.

Therefore the answer is 1/3.

7-2-18

As instructed, we assume that births are independent and the probability of a birth in each day is 1/7. (This is not exactly true; for example, doctors tend to schedule C-sections on weekdays.)

a) The probability that the second person has the same birth day-of-the-week as the first person (whatever that was) is 1/7.

b) We proceed as in Example 13. The probability that all the birth days-of-the-week are different is $p_n = \frac{6}{7} \times \frac{5}{7} ... \frac{8-n}{7}$. since each person after the first must have a different birth day-of-the-week from all the previous people in the group. Note that if $n > 8$, then $p_n = 0$ since the seventh fraction is 0 (this also follows from the pigeonhole principle). The probability that at least two are born on the same day of the week is therefore $1 - p_n$.

c)We compute $1 - p_n$ for $n = 2, 3, , ...$ and find that the first time this exceeds 1/2 is when $n = 4$, so that is our answer. With four people, the probability that at least two will share a birth day-of-the-week is 223/343, or about 65%.

7-2-28

These questions are applications of the binomial distribution. Following the lead of King Henry VIII, we call having a boy success. Then $p = 0.51$ and $n = 5$ for this problem.

a) We are asked for the probability that $k = 3$. By Theorem 2 the answer is $C(5, 3)0.51^3 0.49^2 = 0.32$.

b) There will be at least one boy if there are not all girls. The probability of all girls is 0.49^5, so the answeris $1 - 0.49^5 = 0.972$.

c) This is just like part (b): The probability of all boys is 0.51^5 , so the answer is $1 - 0.51^5 = 0.965$.

d) There are two ways this can happen. The answer is clearly $0.51^5 + 0.49^5 = 0.063$.

7-2-32

Let E be the event that the bit string begins with a 1, and let F be the event that it ends with 00. In each case we need to calculate the probability $p(E \cup F)$, which is the same as $p(E) + p(F) - p(E) \times p(F)$. (The fact that $p(E \cup F) = p(E)p(F)$ follows from the obvious independence of E and F.) So for each part we will compute $p(E)$ and $p(F)$ and then plug into this formula.

a) We have $p(E) = 1/2$ and $p(F) = (1/2)(1/2) = 1/4$. Therefore the answer is $1/2+1/4- = 1/2\times1/4 = 8/5$.

b) We have $p(E) = 0.6$ and $p(F) = (0.4)(0.4) = 0.16$. Therefore the answer is $0.6+0.16-0.6\times0.16 = 0.664$.

c) We have $p(E) = 1/2$ and $P(f) = (1 - \frac{1}{2^9})(1 - \frac{1}{2^{10}})$, thus the answer is $\frac{1}{2} + (1 - \frac{1}{2^9})(1 - \frac{1}{2^{10}}) - \frac{1}{2} \times (1 - \frac{1}{2^9})(1 - \frac{1}{2^{10}}) = 1 - \frac{1}{2^9} + \frac{1}{2^11} + \frac{1}{2^19} - \frac{1}{2^20}$.

d) If a string does not have at least three 1s, then it has 0, 1, or 2 1s. There are $C(10, 0)+C(10, 1)+C(10, 2) = 1 + 10 + 45 = 56$ such strings. There are $2^{10} = 1024$ strings in all. Therefore there are $1024 - 56 = 968$ strings with at least three 1s.

7-3-4

Let F be the event that Ann picks the second box. Thus we know that $p(F) = p(\overline{F}) = 1/2$. Let B be the event that Frida picks an orange ball. Because of the contents of the boxes, we know that $p(B \mid F) = 5/11$ (five of the eleven balls in the second box are orange) and $p(B \mid \overline{F}) = 3/7$. We are asked for $p(F \mid B)$. We use Bayes theorem: $p(F \mid B) = \frac{p(B|F)p(F)}{p(B|F)p(F)+p(B|\overline{F})p(\overline{F})} = 35/68$.

7-3-6

Let S be the event that a randomly chosen soccer player uses steroids. We know that $p(S) = 0.05$ and therefore $p(\overline{S}) = 0.95$. Let P be the event that a randomly chosen person tests positive for steroid use. We are told that $p(P \mid S) = 0.98$ and $p(P \mid \overline{S}) = 0.12$ (this is a false positive test result). We are asked for $p(S \mid P)$. We use Bayes theorem: $p(S \mid P) = \frac{p(P|S)p(S)}{p(P|S)p(S)+p(P|\overline{S})p(\overline{S})} = 0.301$.

7-3-10

Let A be the event that a randomly chosen person in the clinic is infected with avian influenza. We are told that $p(A) = 0.04$ and therefore $p(\overline{A}) = 0.96$. Let P be the event that a randomly chosen person tests positive for avian influenza on the blood test. We are told that $p(P \mid A) = 0.97$ and $p(P \mid \overline{A}) = 0.02$ (false positive). From these we can conclude that $p(\overline{P} \mid A) = 0.03$ (false negative) and $p(\overline{P} \mid \overline{A}) = 0.98$.

a) We are asked for $p(A \mid P)$. We use Bayes theorem: $p(A \mid P) = \frac{p(P|A)p(A)}{p(P|A)p(A)+p(P|\overline{A})p(\overline{A})} = 0.669$.

b)In part (a) we found $p(A \mid P)$. Here we are asked for the probability of the complementary event (given a positive test result). Therefore we have simply $p(\overline{A} \mid P) = 1 - p(A \mid P) = 1 - 0.669 = 0.331$

c)We are asked for $p(A \mid \overline{P})$. We use Bayes theorem: $p(A \mid \overline{P}) = \frac{p(\overline{P}|A)p(A)}{p(\overline{P}|A)p(A)+p(\overline{P}|\overline{A})p(\overline{A})} = 0.001$.

d)In part (c) we found $p(A \mid \overline{P})$. Here we are asked for the probability of the complementary event (given a negative test result). Therefore we have simply $p(\overline{A} \mid \overline{P}) = 1 - p(A \mid \overline{P}) = 1 - 0.001 = 0.999$.

7-4-2

By Theorem 2 the expected number of successes for n Bernoulli trials is np. In the present problem we have $n = 10$ and $p = 1/2$. Therefore the expected number of successes (i.e., appearances of a head) is $10 \times (1/2) = 5$.

7-4-6

There are $C(50, 6)$ equally likely possible outcomes when the state picks its winning numbers. The probability of winning 10 million is therefore $1/C(50, 6)$, and the probability of winning 0 is $1 - (1/C(50, 6))$. By definition, the expectation is therefore $10,000,000 \times 1/C(50, 6) + 0 = 0.63$

7-4-8

By Theorem 3 we know that the expectation of a sum is the sum of the expectations. In the current exercise we can let X be the random variable giving the value on the first die, let Y be the random variable giving the value on the second die, and let Z be the random variable giving the value on the third die. In order to compute the expectation of X, of Y , and of Z , we can ignore what happens on the dice not under consideration. Looking just at the first die, then, we compute that the expectation of X is $1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 3.5$.
Similarly, $E(Y) = 3.5$ and $E(Z) = 3.5$. Therefore $E(X + Y + Z) = 3 \times 3.5 = 10.5$.

7-4-10

There are 6 different outcomes of our experiment. Let the random variable X be the number of times we flipthe coin. For $i = 1, 2, ..., 6$, we need to compute the probability that $X = i$. In order for this to happen when $i < 6$, the first $i - 1$ flips must contain exactly one tail, and there are $i - 1$ ways this can happen. Therefore $p(X = i) = (i - 1)/2i$, since there are $2i$ equally likely outcomes of i flips. So we have $p(X = 1) = 0$, $p(X = 2) = 1/4$, $p(X = 3) = 2/8 = 1/4$, $p(X = 4) = 3/16$, $p(X = 5) = 1/8$. To compute $p(X = 6)$, we note that this will happen when there is exactly one tail or no tails among the first five flips (probability $5/32 + 1/32 = 3/16$). We compute the expected number by summing i times $p(X = i)$, so we get $1 \times 0 + 2 \times 1/4 + 3 \times 1/4 + 4 \times 3/16 + 5 \times 1/8 + 6 \times 3/16 = 3.75$.

7-4-12

If X is the number of times we roll the die, then X has a geometric distribution with $p = 1/6$.
a)$p(X = n) = (1 - p)^{n-1}p = (5/6)^{n-1}(1/6) = 5^{n-1}/6^n$.
b)$1/(1/6) = 6$ by Theorem 4.