# CS271 Homework 5 Solution

**6-1-12**
We use the sum rule, adding the number of bit strings of each length up to 6. If we include the empty string, then we get $2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 2^7 - 1 = 127$.

**6-1-18**
Recall that a DNA sequence is a sequence of letters, each of which is one of $A$, $C$, $G$, or $T$. Thus by the product rule there are $4^5 = 1024$ DNA sequences of length five if we impose no restrictions. **a)** If the sequence must end with $A$, then there are only four positions at which to make a choice, so the answer is $4^4 = 256$.
**b)** If the sequence must start with $T$ and end with $G$,
then there are only three positions at which to make achoice, so the answer is $4^3 = 64$.
**c)** If only two letters can be used rather than four, the number of choices is $2^5 = 32$.
**d)** As in part (c), there are $3^5 = 243$ sequences that do not contain $C$.

**6-1-30**
$26^3 10^3 + 26^4 10^2 = 63,273,600$

**6-2-2**
This follows from the pigeonhole principle, with $k = 26$

**6-2-14**
**a)** We can group the first ten positive integers into five subsets of two integers each, each subset adding up to 11: $\{1, 10\}$,$\{2, 9\}$, $\{3, 8\}$, $\{4, 7\}$, and $\{5, 6\}$. If we select seven integers from this set, then by the pigeonhole principle at least two of them come from the same subset. Furthermore, if we forget about these two in the same group, then there are five more integers and four groups; again the pigeonhole principle guarantees two integers in the same group. This gives us two pairs of integers, each pair from the same group. In each case these two integers have a sum of 11, as desired.
**b)** No. The set $\{1, 2, 3, 4, 5, 6\}$ has only 5 and 6 from the same group, so the only pair with sum 11 is 5 and 6.

**6-3-10**
$P(6, 6) = 6! = 720$

**6-3-20**

**a)** There are $C(10,3)$ ways to choose the positions for the 0s, and that is the only choice to be made, so the answer is $C(10,3)=120$.
**b)** There are more 0s than 1s if there are fewer than five 1s. Using the same reasoning as in part (a), together with the sum rule, we obtain the answer $C(10,0)+C(10,1)+C(10,2)+C(10,3)+C(10,4)=1+10+45+120+210=386$. Alternatively, by symmetry, half of all cases in which there are not five 0s have more 0s than 1s; therefore the answer is $(2^{10}-C(10,5))/2=(1024-252)/2=386$.
**c)** We want the number of bit strings with 7, 8, 9, or 10 1s. By the same reasoning as above, there are $C(10,7)+C(10,8)+C(10,9)+C(10,10)=120+45+10+1=176$ such strings.
**d)** If a string does not have at least three 1s, then it has 0, 1, or 2 1s. There are $C(10,0)+C(10,1)+C(10,2)=1+10+45=56$ such strings. There are $2^{10}=1024$ strings in all. Therefore there are $1024-56=968$ strings with at least three 1s.

**6-3-26**
**a)** This is just a matter of choosing 10 players from the group of 13, since we are not told to worry about what positions they play; therefore the answer is $C(13,10)=286$.
**b)** This is the same as part (a), except that we need to worry about the order in which the choices are made, since there are 10 distinct positions to be filled. Therefore the answer is $P(13,10)=13!/3!=1,037,836,800$.
**c)** There is only one way to choose the 10 players without choosing a woman, since there are exactly 10 men. Therefore (using part (a)) there are $286-1=285$ ways to choose the players if at least one of them must be a woman.

**6-2-32**
**a)** The only reasonable way to do this is by subtracting from the number of strings with no restrictions the number of strings that do not contain the letter $a$. The answer is $26^6-25^6=308915776-244140625=64,775,151$.
**b)** If our string is to contain both of these letters, then we need to subtract from the total number of strings the number that fail to contain one or the other (or both) of these letters. As in part (a), $25^6$ strings fail to contain an $a$; similarly $25^6$ fail to contain a $b$. This is overcounting, however, since $24^6$ fail to contain both of these letters. Therefore there are $25^6+25^6-24^6$ strings that fail to contain at least one of these letters. Therefore the answer is $26^6-(25^6+25^6-24^6)=308915776-(244140625+244140625-191102976)=11,737,502$.
**c)** First choose the position for the $a$; this can be done in 5 ways, since the $b$ must follow it. There are four remaining positions, and these can be filled in $P(24,4)$ ways, since there are 24 letters left (no repetitions being allowed this time). Therefore the answer is $5P(24,4)=1,275,120$.
**d)** First choose the positions for the $a$ and $b$; this can be done in $C(6,2)$ ways, since once we pick two positions, we put the $a$ in the left-most and the $b$ in the other. There are four remaining positions, and these can be filled in $P(24,4)$ ways, since there are 24 letters left (no repetitions being allowed this time). Therefore the answer is $C(6,2)P(24,4)=3,825,360$.

**6-4-8**
$\binom{17}{9}3^8 2^9=24310\cdot 6561\cdot 512=81,662,929,920$

**6-4-12**
We just add adjacent numbers in this row to obtain the next row (starting and ending with 1, of course):
1 11 55 165 330 462 462 330 165 55 11 1

**6-5-10**
**a)** $C(6+12-1,12) = C(17,12) = 6188$
**b)** $C(6+36-1,36) = C(41,36) = 749,398$.
**c)** If we first pick the two of each kind, then we have picked $2 \cdot 6 = 12$ croissants. This leaves one dozen left to pick without restriction, so the answer is the same as in part (a), namely $C(6+12-1,12) = C(17,12) = 6188$.
**d)** We first compute the number of ways to violate the restriction, by choosing at least three broccoli croissants. This can be done in $C(6+21-1,21) = C(26,21) = 65780$ ways, since once we have picked the three broccoli croissants there are 21 left to pick without restriction. Since there are $C(6+24-1,24) = C(29,24) = 118755$ ways to pick 24 croissants without any restriction, there must be $118755 - 65780 = 52,975$ ways to choose two dozen croissants with no more than two broccoli.
**e)** Eight croissants are specified, so this problem is the same as choosing $24 - 8 = 16$ croissants without restriction, which can be done in $C(6+16-1,16) = C(21,16) = 20,349$ ways.
**f)** First let us include all the lower bound restrictions. If we choose the required 9 croissants, then there are $24 - 9 = 15$ left to choose, and if there were no restriction on the broccoli croissants then there would be $C(6+15-1,15) = C(20,15) = 15504$ ways to make the selections. If in addition we were to violate the broccoli restriction by choosing at least four broccoli croissants, there would be $C(6+11-1,11) = C(16,11) = 4368$ choices. Therefore the number of ways to make the selection without violating the restriction is $15504 - 4368 = 11,136$.

**6-5-18**
It follows directly from Theorem 3 that the answer is $\frac{20!}{2!4!3!1!2!3!2!3!} = 5.9 \times 10^{13}$.

**6-5-24**
We assume that this problem leaves us free to pick which boxes get which numbers of balls. There are several ways to count this. Here is one. Line up the 15 objects in a row (15! ways to do that), and line up the five boxes in a row (5! ways to do that). Now put the first object into the first box, the next two into the second box, the next three into the third box, and so on. This overcounts by a factor of $1! \cdot 2! \cdot 3! \cdot 4! \cdot 5!$, since there are that many ways to swap objects in the permutation without affecting the result. Therefore the answer is $15! \cdot 5!/(1! \cdot 2! \cdot 3! \cdot 4! \cdot 5!) = 4,540,536,000$.

**6-5-30**
By Theorem 3 the answer is $11!/(4!4!2!) = 34,650$.

**6-5-34**
We need to calculate separately, using Theorem 3, the number of strings of length 5, 6, and 7. There are $7!/(3!3!1!) = 140$ strings of length 7. For strings of length 6, we can omit the $R$ and form $6!/(3!3!) = 20$ strings; omit an $E$ and form $6!/(3!2!1!) = 60$ strings, or omit an $S$ and also form 60 strings. This gives a total of 140 strings of length 6. For strings of length 5, we can omit two $E$s or two $S$s, each giving $5!/(3!1!1!) = 20$ strings; we can omit one $E$ and one $S$ ($5!/(2!2!1!) = 30$ strings); or we can omit the $R$ and either an $E$ or an $S$ ($5!/(3!2!) = 10$ strings each). This gives a total of 90 strings of length 5, for a grand total of 370 strings of length 5 or greater.

**6-5-38**
We assume that the forty issues are distinguishable.
**a)** Theorem 4 says that the answer is $40!/10!4 = 4.7 \times 10^{21}$ .

**b)** Each distribution into identical boxes gives rise to $4! = 24$ distributions into labeled boxes, since once we have made the distribution into unlabeled boxes we can arbitrarily label the boxes. Therefore the answer is the same as the answer in part (a) divided by 24, namely $(40!/10!4)/4! = 2.0 \times 10^{20}$ .

**6-6-8**

The first subset corresponds to the bit string 0000, namely the empty set. The next subset corresponds to the bit string 0001, namely the set $\{4\}$. The next bit string is 0010, corresponding to the set $\{3\}$, and then 0011, which corresponds to the set $\{3, 4\}$. We continue in this manner, giving the remaining sets: $\{2\}$, $\{2, 4\}$, $\{2, 3\}$, $\{2, 3, 4\}$, $\{1\}$, $\{1, 4\}$, $\{1, 3\}$, $\{1, 3, 4\}$, $\{1, 2\}$, $\{1, 2, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 3, 4\}$.