CS271 Homework 3 Solution

5-1-10

a) By computing the first few sums and getting the answers 1/2, 2/3, and 3/4, we guess that the sum is n/(n + 1).

b) We prove this by induction. It is clear for n = 1, since there is just one term, 1/2. Suppose that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

We want to show that

$$[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... + \frac{1}{k(k+1)}] + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

Starting from the left, we replace the quantity in brackets by k/(k + 1) (by the inductive hypothesis), and then do the algebra

$$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

5-1-14

We proceed by induction. Notice that the letter k has been used in this problem as the dummy index of summation, so we cannot use it as the variable for the inductive step. We will use n instead. For the basis step we have $1 \cdot 2^1 = (1-1)2^{1+1} + 2$, which is the true statement $2 = 2$. We assume the inductive hypothesis, that

$$\sum_{k=1}^{n} k \cdot 2^k = (n-1)2^{n+1} + 2$$

and try to prove that

$$\sum_{k=1}^{n+1} k \cdot 2^k = n2^{n+2} + 2$$

Splitting the left-hand side into its first n terms followed by its last term and invoking the inductive hypothesis, we have

$$\sum_{k=1}^{n} k \cdot 2^k = \sum_{k=1}^{n} k \cdot 2^k + (n+1)2^{n+1} = (n-1)2^{n+1} + 2 + (n+1)2^{n+1} = 2n \cdot 2^{n+1} + 2 = n2^{n+2} + 2$$

as desired.

5-1-28

The base case is $n = 3$. We check that $4^2 - 7 \cdot 4 + 12 = 0$ is nonnegative. Next suppose that $n^2 - 7n + 12 \geq 0$; we must show that $(n+1)^2-7(n+1)+12 \geq 0$. Expanding the left-hand side, we obtain $n^2+2n+1-7n-7+12 = (n^2-7n+12)+(2n-6)$. The first of the parenthesized expressions is nonnegative by the inductive hypothesis; the second is clearly also nonnegative by the assumption that n is at least 3. Therefore their sum is nonnegative, and the inductive step is complete.

5-2-6

a) We can form the following amounts of postage as indicated: $3 = 3$, $6 = 3 + 3$, $9 = 3 + 3 + 3$, $10 = 10$, $12 = 3 + 3 + 3 + 3$, $13 = 10 + 3$, $15 = 3 + 3 + 3 + 3 + 3$, $16 = 10 + 3 + 3$, $18 = 3 + 3 + 3 + 3 + 3 + 3$, $19 = 10 + 3 + 3 + 3$, $20 = 10 + 10$. By having considered all the combinations, we know that the gaps in this list cannot be filled. We claim that we can form all amounts of postage greater than or equal to 18 cents using just 3-cent and 10-cent stamps.

b) Let $P(n)$ be the statement that we can form n cents of postage using just 3-cent and 10-cent stamps. We want to prove that $P(n)$ is true for all $n \geq 18$. The basis step, $n = 18$, is handled above. Assume that we can form k cents of postage (the inductive hypothesis); we will show how to form $k + 1$ cents of postage. If the k cents included two 10-cent stamps, then replace them by seven 3-cent stamps ($7 \cdot 3 = 2 \cdot 10 + 1$). Otherwise, k cents was formed either from just 3-cent stamps, or from one 10-cent stamp and $k - 10$ cents in 3-cent stamps. Because $k \geq 18$, there must be at least three 3-cent stamps involved in either case. Replace three 3-cent stamps by one 10-cent stamp, and we have formed $k + 1$ cents in postage ($10 = 3 \cdot 3 + 1$).

c) $P(n)$ is the same as in part (b). To prove that $P(n)$ is true for all $n \geq 18$, we note for the basis step that from part (a), $P(n)$ is true for $n = 18, 19, 20$. Assume the inductive hypothesis, that $P(j)$ is true for all j with $18 \geq j \geq k$, where k is a fixed integer greater than or equal to 20. We want to show that $P(k + 1)$ is true. Because $k - 2 \geq 18$, we know that $P(k - 2)$ is true, that is, that we can form $k - 2$ cents of postage. Put one more 3-cent stamp on the envelope, and we have formed $k + 1$ cents of postage, as desired. In this proof our inductive hypothesis included all values between 18 and k inclusive, and that enabled us to jump back three steps to a value for which we knew how to form the desired postage.

5-2-10

We claim that it takes exactly $n - 1$ breaks to separate a bar (or any connected piece of a bar obtained by horizontal or vertical breaks) into n pieces. We use strong induction. If $n = 1$, this is trivially true (one piece, no breaks). Assume the strong inductive hypothesis, that the statement is true for breaking into k or fewer pieces, and consider the task of obtaining k + 1 pieces. We must show that it takes exactly k breaks. The process must start with a break, leaving two smaller pieces. We can view the rest of the process as breaking one of these pieces into $i + 1$ pieces and breaking the other piece into $k - i$ pieces, for some i between 0 and $k-1$, inclusive. By the inductive hypothesis it will take exactly i breaks to handle the first piece and $k-i-1$ breaks to handle the second piece. Therefore the total number of breaks will be $1 + i + (k - i - 1) = k$, as desired.

5-2-32

The proof is invalid for $k = 4$. We cannot increase the postage from 4 cents to 5 cents by either of the replacements indicated, because there is no 3-cent stamp present and there is only one 4-cent stamp present. There is also a minor flaw in the inductive step, because the condition that $j \geq 3$ is not mentioned.

5-3-8

Many answers are possible.

a) Each term is 4 more than the term before it. We can therefore define the sequence by $a_1 = 2$ and $a_{n+1} = a_n + 4$ for all $n \geq 1$.

b) We note that the terms alternate: 0, 2, 0, 2, and so on. Thus we could define the sequence by $a_1 = 0$, $a_2 = 2$, and $a_n = a_{n-2}$ for all $n \geq 3$.
c) The sequence starts out 2, 6, 12, 20, 30, and so on. The differences between successive terms are 4, 6, 8, 10, and so on. Thus the nth term is $2n$ greater than the term preceding it; in symbols: $a_n = a_{n-1} + 2n$. Together with the initial condition $a_1 = 2$, this defines the sequence recursively.
d) The sequence starts out 1, 4, 9, 16, 25, and so on. The differences between successive terms are 3, 5, 7, 9, and so onthe odd numbers. Thus the nth term is $2n - 1$ greater than the term preceding it; in symbols: $a_n = a_{n-1} + 2n - 1$. Together with the initial condition $a_1 = 1$, this defines the sequence recursively.

5-3-22

Clearly only positive integers can be in S, since 1 is a positive integer, and the sum of two positive integers is again a positive integer. To see that all positive integers are in S, we proceed by induction. Obviously $1 \in S$. Assuming that $n \in S$, we get that $n + 1$ is in S by applying the recursive part of the definition with $s = n$ and $t = 1$. Thus S is precisely the set of positive integers.

5-4-8

The sum of the first n positive integers is the sum of the first $n - 1$ positive integers plus n. This trivial observation leads to the recursive algorithm shown here.

```
proc sum(n)
  if n = 1 then return 1
           else return sum(n − 1) + n
```

5-4-24

We use the hint.

```
proc twopower(n, a)
  if n = 1 then return a^2
           else return twopower(n − 1, a)^2
```

5-4-46

From the analysis given before the statement of Lemma 1, it follows that the number of comparisons is $m+n-r$, where the lists have m and n elements, respectively, and r is the number of elements remaining in one list at the point the other list is exhausted. In this exercise $m = n = 5$, so the answer is always $10 - r$.
a) The answer is $10 - 1 = 9$, since the second list has only 1 element when the first list has been emptied.
b) The answer is $10 - 5 = 5$, since the second list has 5 elements when the first list has been emptied.
c) The answer is $10 - 2 = 8$, since the second list has 2 elements when the first list has been emptied.